DIFFERENTIAL EQUATIONS

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1 ORDINARY DIFFERENTIAL EQUATIONS

1.1 Order

D.E.'s are classified in orders, depending on the highest-order derivative present:

$$\frac{dy}{dx} = 2y \qquad \text{is 1st order}$$
$$y\ sin\ x + 2\frac{dy}{dx} = 3 \qquad \text{is 1st order}$$
$$(\frac{dy}{dx})^3 + (\frac{dy}{dx})^2 = -x \qquad \text{is 1st order}$$
$$1 + (\frac{dy}{dx})^3 = \frac{d^2y}{dx^2} \qquad \text{is 2nd order}$$
$$\frac{d^2y}{dx^2} + n^2x = 0 \qquad \text{is 2nd order}$$
$$x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad \text{is 3rd order}$$

1.2 Elimination of arbitrary constants

D.E.'s can be formed by the elimination of arbitrary constants from a solution:

eg: if $y = a\ sin\ x + b\ cos\ x$ (1)

then differentiating, $dy/dx = a\ cos\ x - b\ sin\ x$

and again, $d^2y/dx^2 = -a\ sin\ x - b\ cos\ x$ (2)

adding (1) + (2), $\dfrac{d^2y}{dx^2} + y = 0$ ✓

or: if $y = e^{-x}(a\ sin\ x + b\ cos\ x)$

we let $u = ye^x$

so that $d^2u/dx^2 + u = 0$ (3)

but $\dfrac{du}{dx} = \dfrac{d(ye^x)}{dx} = ye^x + e^x\dfrac{dy}{dx}$

moreover $\dfrac{d^2u}{dx^2} = ye^x + 2e^x\dfrac{dy}{dx} + e^x\dfrac{d^2y}{dx^2}$

which, from (3) $= -ye^x$

simplifying $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$ ✓

or: if	$y = a + bx + cx^2 + dx^3$
differentiating	$dy/dx = b + 2cx + 3dx^2$
and again	$d^2y/dx^2 = 2c + 6dx$
and again	$d^3y/dx^3 = 6d$
and again	$d^4y/dx^4 = 0$ ✓

1.3 General solution and particular solution

The 'General Solution' of an n'th order D.E. contains n arbitrary constants, and satisfies the equation.

A 'Particular Solution' is *any* function which satisfies the equation. Also known as a 'Particular Integral'.

1.4 Raimes' Rule for the solution of differential equations

Find the solution by hook or by crook.

1.5 Separable D.E.'s of the 1st order

These are D.E.'s of the form $dy/dx = F(x) \cdot G(y)$ and can be solved as follows:

separate the variables $$\frac{dy}{G(y)} = F(x) \cdot dx$$

and integrate $$\int \frac{dy}{G(y)} = \int F(x) \cdot dx$$

eg: (a trivial case) $$dy/dx = f(x) \cdot 1$$

gives $$y = \int f(x) dx + c \quad ✓$$

or: (a trivial case) $$dy/dx = 1 \cdot g(y)$$

gives $$y = \int \frac{dy}{g(y)} + c \quad ✓$$

or $$dy/dx = x \cdot y$$

gives $$log\ y = x^2/2 + c$$

i.e. $$y = A \cdot exp(x^2/2) \quad ✓$$

or $$dy/dx = e^{3x-y}$$

gives $$exp(y) = (1/3) \cdot exp(3x) + c \quad ✓$$

or $$dy/dx = 2x \cdot sec(y)$$

gives $$sin(y) = x^2 + c \quad ✓$$

1.6 Homogeneous D.E.'s of the 1st order

These are D.E.'s of the form $dy/dx = f(y/x)$ and can be solved as follows:

given $$\frac{dy}{dx} = f(\frac{y}{x})$$
we let $$y = v \cdot x$$
so that $$dy/dx = v + x \cdot dv/dx = f(v)$$
i.e. $$x \cdot dv/dx = f(v) - v$$ which is separable,
i.e. $$\frac{dx}{x} = \frac{dv}{f(v) - v}$$
and integrating $$log(Ax) = \int \frac{dv}{f(v) - v}$$ ✓

eg: $$\begin{aligned} dy/dx &= (x^2 + y^2)/xy \\ &= (1 + (y/x)^2)/(y/x) \end{aligned}$$
we let $$y = vx$$
so that $$\begin{aligned} dy/dx &= (1 + v^2)/v \\ &= 1/v + v \end{aligned}$$
therefore $$v + x \cdot dv/dx = 1/v + v$$
i.e. $$x \cdot dv/dx = 1/v$$
i.e. $$log(Ax) = v^2/2$$
i.e. $$y^2 = 2x^2 log(Ax)$$ ✓

1.7 D.E.'s reducible to homogeneous form by a substitution

Consider $$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

We can solve this as follows, since it is homogeneous except for c and c' . . .

Substitute $$u = ax + by + c$$
and $$v = a'x + b'y + c'$$
which gives $$du/dx = a + b(dy/dx)$$
and $$dv/dx = a' + b'(dy/dx)$$
dividing $$du/dv = (a + b(dy/dx))/(a' + b'(dy/dx))$$
but $$dy/dx = u/v$$
so we have $$du/dv = (a + b(u/v))/(a' + b'(u/v))$$ which is homogeneous.
so substitute $$w = u/v$$
or $$u = vw$$
i.e. $$du/dv = w + v(dw/dv)$$ as in the previous section

For example $$dy/dx = (2x + 2y - 2)/(3x + y - 5)$$
we let $$u = 2x + 2y - 2 \tag{4}$$
and $$v = 3x + y - 5 \tag{5}$$
which gives $$du/dx = 2 + 2(dy/dx)$$
and $$dv/dx = 3 + dy/dx$$
dividing $$\begin{aligned} du/dv &= (2 + 2(dy/dx))/(3 + dy/dx) \\ &= (2 + 2u/v)/(3 + u/v) \end{aligned}$$ which is homogeneous
so we let $$w = u/v \tag{6}$$
or $$u = vw$$

$$\begin{aligned}
&\text{i.e.} && du/dv = w + v(dw/dv) \\
&\text{but this} && = (2+2w)/(3+w)) \\
&\text{therefore} && v(dw/dv) = (2-w-w^2)/(3+w)) \\
&\text{integrating} && log\ Av = \int \frac{3+w}{(2+w)(1-w)} dw \\
&\text{which gives} && log\ Av = (log(2+w) - 4\ log(1-w))/3 \\
&\text{i.e.} && av^3 = (2+w)/(1-w))^4 \quad \checkmark
\end{aligned}$$

Then we would substitute back using equations (6), (5) and (4).
Note that a simpler case arises if $a'/a = b'/b = m$

$$\begin{aligned}
&\text{so that} && dy/dx = (ax+by+c)/(m(ax+by)+c') \\
&\text{which} && = (ax+by+c)/(m(ax+by+c)+c'-mc) \\
&\text{which} && = f(ax+by+c) \\
&\text{so , let} && v = ax+by+c \\
&\text{so that} && dv/dx = a + b(dy/dx) \\
&\text{or} && dv/dx = a + b \cdot f(v) \qquad \text{which is already separable.}
\end{aligned}$$

1.8 Linear D.E.'s of the 1st order

These are D.E.'s of the form : $dy/dx + P(x) \cdot y = Q(x)$

Simple cases can be solved directly:

$$\begin{aligned}
&\text{e.g.} && dy/dx + y/x = x^2 \\
&\text{times x} && x \cdot dy/dx + y = x^3 \\
&\text{which is !} && d(xy)/dx = x^3 \\
&\text{integrating} && xy = x^4/4 + c \quad \checkmark
\end{aligned}$$

In general, they can be solved by multiplying by an 'integrating factor' $exp(\int Pdx)$ (see 1.16)

$$\begin{aligned}
&\text{first note that} && d\left(exp\left(\int Pdx\right)\right)/dx = exp\left(\int Pdx\right) \cdot d\left(exp\left(\int Pdx\right)\right)/dx \\
& && = P\ exp\left(\int Pdx\right) \\
&\text{The D.E becomes} && exp\left(\int Pdx\right) \cdot (dy/dx + Py) = Q\ exp\left(\int Pdx\right) \\
&\text{which is !} && d\left(y \cdot exp\left(\int Pdx\right)\right)/dx = Q\ exp\left(\int Pdx\right) \\
&\text{integrating} && y \cdot exp\left(\int Pdx\right) = \int Q\ exp\left(\int Pdx\right)\ dx + c \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
&\text{e.g.} && sin(x) \cdot dy/dx + y\ cos(x) = sin^2\ x \\
&\text{i.e.} && dy/dx + y\ cot(x) = sin(x) \\
&\text{the integrating factor is} && exp\left(\int cot\ x\right) = exp(log(sin\ x)) = sin\ x \\
&\text{multiplying through} && sin\ x \cdot dy/dx + y\ cos\ x = sin^2\ x \qquad \text{(blush !)} \\
&\text{which gives} && y \cdot sin\ x = \int sin^2\ x\ dx + c \\
& && = (2x - sin\ 2x)/4 \quad \checkmark
\end{aligned}$$

e.g. $\quad dy/dx = (x^2 - y)\ x$

i.e. $\quad dy/dx + xy = x^3$

the integrating factor is $\quad exp\left(\int x dx\right) = exp(\frac{x^2}{2})$

multiplying though gives $\quad y \cdot exp(\frac{x^2}{2}) = \int x^3 \cdot exp(\frac{x^2}{2})\ dx + c$

we can solve this with $\quad u = x^2/2$

so that $\quad du = x\ dx$

thus $\quad \int x^3 \cdot exp(\frac{x^2}{2}) \cdot dx = \int 2u \cdot e^u du$

by parts $\quad = 2u \cdot e^u - \int 2e^u du$

$\quad = 2u \cdot e^u - 2e^u$

therefore $\quad y \cdot exp(x^2/2) = (x^2 - 2)\ exp(x^2/2) + c \quad$ ✓

1.9 Bernouilli's Equation

This is a D.E. of the form $\quad dy/dx + P(x) \cdot y = Q(x) \cdot y^n \quad$ We solve it as follows:

divide by y^n $\quad y^{-n} \cdot dy/dx + P(x) \cdot y^{1-n} = Q(x)$

and now let $\quad z = y^{1-n}$

so that $\quad dz/dx = (1-n) \cdot y^{-n} \cdot dy/dx$

substituting, $\quad dz/dx + (1-n) \cdot P(x) \cdot z = (1-n) \cdot Q(x)$

This is now a linear D.E. of the 1st order, and we solve it as in Section 1.8.

e.g. $\quad x^2 y - x^3\ dy/dx = y^4 \cdot cos(x)$

rearrange $\quad dy/dx - y/x = -y^4 \cdot (cos\ x)/x^3$

divide by y^n $\quad y^{-4} \cdot dy/dx - y^{-3}/x = (cos\ x)/x^3$

now let $\quad z = y^{-3} \quad (7)$

so that $\quad dz/dx = -3 \cdot y^{-4} \cdot dy/dx$

substituting, $\quad dz/dx + 3z/x = 3(cos\ x)/x^3 \quad$ this is linear of the 1st order

the integrating factor is $\quad exp\left(\int \frac{3}{x}\ dx\right) = x^3$

therefore $\quad z\ x^3 = \int 3\ cos\ x \cdot \frac{x^3}{x^3}\ dx$

integrating, $\quad z\ x^3 = 3\ sin\ x + c$

substituting (7), $\quad y^3 = x^3/(3\ sin\ x + c) \quad$ ✓

1.10 D.E's of the 2nd order with Constant Coefficients

These are D.E.'s of the form

$$d^2y/dx^2 + A\ dy/dx + B\ y = f(x) \tag{8}$$

where A and B are constants. If $f \equiv 0$, the D.E is called the 'homogeneous equation'

$$d^2y/dx^2 + A\ dy/dx + B\ y = 0$$

The general solution of the homogeneous equation is called the 'complimentary function', or 'C.F.' and suppose that $y = v(x)$ is a 'particular integral', or 'P.I.', of (8).

Let us put $C.F. = u(x)$
and $P.I. = v(x)$
so we have $d^2v/dx^2 + A\ dv/dx + B\ v = f(x)$
and $d^2u/dx^2 + A\ du/dx + B\ u = 0$
adding, $d^2(u+v)/dx^2 + A\ d(u+v)/dx + B\ (u+v) = f(x)$

Therefore $(u + x)$ is a solution of (8); and it is the general solution since it contains two arbitrary constants in u. Therefore

$$general\ solution = complimentary\ function + particular\ integral$$

Thus solving these equations is done in two halves . . .

1.10.1 Finding the Complimentary Function

This is the general solution of $$d^2y/dx^2 + A\ dy/dx + B\ y = 0 \tag{9}$$

To simplify notation, we introduce the operator $D = d/dx$ so (9) becomes

$$(D^2 + AD + B)\ y = 0$$

To solve this, we factorise this into $$(D - a)(D - b) = 0 \tag{10}$$
this has two solutions $D = a$ and $D = b$
i.e. $dy/y = a\ dx$ and $dy/y = b\ dx$
integrating $log\ y = ax + c$ and $log\ y = bx + c$
i.e. $y = C_1\ e^{ax}$ and $y = C_2 e^{bx}$
Therefore the sum $$y = C_1\ e^{ax} + C_2 e^{bx} \tag{11}$$

is also a solution, and indeed **if a is not equal to b** then it has two arbitrary constants, and is therefore the general solution.

a and b are the roots of $m^2 + Am + B = 0$ which is known as the 'auxiliary equation'.

But if a and b are equal then (11) becomes $y = C_3 e^{ax}$ which has only one arbitrary constant, and is thus not a general solution. We can find the general solution for this case with the substitution:

let $y = v\ e^{ax}$
so that $Dy = D(v\ e^{ax}) = e^{ax}\ Dv + a \cdot v \cdot e^{ax}$
$= e^{ax} \cdot (D + a) \cdot v$
so that $(D - a)^2(v\ e^{ax}) = (D - a)(e^{ax} \cdot Dv)$
$= e^{ax} \cdot D^2 v$
so that (10) becomes $e^{ax} \cdot D^2 v = 0$
i.e. $D^2 v = 0$
integrating $Dv = c_2$
integrating again $v = c_2 x + c_1$
substituting, the general solution is $y = (c_1 + c_2 x) \cdot e^{ax}$ ✓

e.g. $D^2 y - y = 0$
i.e. $(D^2 - 1)\ y = 0$
the auxiliary equation is $m^2 - 1 = 0$
this has two distinct roots $m = 0 \pm 1$ i.e. $a = 1$ $b = -1$
therefore the c.f is $y = c_1 e^{+x} + c_2 e^{-x}$ ✓

or e.g. $D^2y + 3Dy - 4y = 0$

i.e. $(D^2 + 3D - 4)\ y = 0$

the auxiliary equation is $m^2 + 3m - 4 = 0$

this has two distinct roots $m = -4, +1$ i.e. $a = 1$ $b = -4$

therefore the c.f is $y = c_1 e^x + c_2 e^{-4x}$ ✓

or e.g. $d^2y/dx^2 + 6dy/dx + 9y = 0$

i.e. $(D^2 + 6D + 9)\ y = 0$

auxiliary equation $m^2 + 6m + 9 = 0$

this has two equal roots ! $a = -3, \quad b = -3$

thus the c.f is $y = (c_1 + c_2 x)\ e^{-3x}$ ✓

or e.g. $d^2y/dx^2 + 4y = 0$

i.e. $(D^2 + 4)\ y = 0$

the roots are $a = 2i, \quad b = -2i$

thus the c.f is

$$
\begin{aligned}
y &= c_1 e^{2ix} + c_2 e^{-2ix} \\
&= c_1(cos2x + i\ sin2x) + c_2(cos2x - i\ sin2x) \\
&= (c_1 + c_2)\ cos\ 2x + i(c_1 - c_2)\ sin\ 2x \\
&= c_3\ cos\ 2x + c_4\ sin\ 2x
\end{aligned}
$$
✓

or e.g. $d^2y/dx^2 - 2dy/dx + 3y = 0$

i.e. $(D^2 + -2D + 3)\ y = 0$

auxiliary equation $m^2 - 2m + 3 = 0$

the roots are

$$
\begin{aligned}
m &= (2 \pm \sqrt{-8})/2 \ = \ 1 \pm i\sqrt{2} \\
&= c_1\ exp((1 + i\sqrt{2})x) + c_2\ exp((1 - i\sqrt{2})x) \\
&= e^x(c_1\ exp(i\sqrt{2}x) + c_2\ exp(-i\sqrt{2}x)) \\
&= e^x(A\ cos(\sqrt{2}x) + B\ sin(\sqrt{2}x)
\end{aligned}
$$
✓

1.10.2 Finding the Particular Integral if $f(x)$ is a Constant

If $f(x)$ is a constant, $d^2x/dy^2 + Ady/x + By = c$

then it's easy : $y = c$

1.10.3 Finding the Particular Integral if $f(x)$ is a Polynomial

Before we tackle this, we must digress to define **the inverse of an operator**.

D^{-1} is defined to be such that $D \cdot D^{-1} \cdot y \equiv y$

so that if $D \equiv d/dx$, then $D^{-1} = \int (\ )dx$

e.g. $x/D = x^2/2$

or $D^{-2}e^{cx} = c^{-2}e^{cx}$

Similarly, $(D - a)^{-1}$ is defined by : $(D - a) \cdot (D - a)^{-1} \cdot y \equiv y$

thus our equation $(D^2 + AD + B)\ y = f(x)$
i.e. $(D-a)(D-b)\ y = f(x)$
which we write as $y = f(x)\ /\ ((D-a)(D-b))$

e.g. consider $(D^2 - 3D + 2)\ y = x$ $\quad$ x is a simple polynomial !
or $(D-2)(D-1)\ y = x$
or

$$\begin{aligned} y &= x\ /\ ((D-2)(D-1)) \\ &= (1/2)\ /\ ((1-D/2)(1-D)) \cdot x \\ &= \frac{1}{2}(1+\frac{D}{2}+\frac{D^2}{4}+\frac{D^3}{8}+..)(1+D+D^2+D^3+..)\cdot x \\ &= \frac{1}{2}(1+\frac{D}{2}+\frac{D^2}{4}+\frac{D^3}{8}+..)(1+x+0+0+0+..) \\ &= \frac{1}{2}(1+x+\frac{1}{2}+\frac{0}{4}+\frac{0}{8}+...) \\ &= x/2+3/4 \qquad \text{which is the particular integral} \end{aligned}$$

the general solution is $y = Ae^{2x} + Be^{x} + x/2 + 3/4$ ✓

or consider $(D^2 + 2D)\ y = x^2$
factorising $D\ (D+2)\ y = x^2$
thus the P.I. is

$$\begin{aligned} y &= (1\ /\ (D(D+2))\cdot x^2 \\ &= (1/2D)\ (1 - D/2 + D^2/4 + D^3/8 + ...)\cdot x^2 \\ &= (1/2D)\ (x^2 - x + 1/2) \\ &= (1/2)\ (x^3/3 - x^2/2 + x/2) \qquad \text{(check by substitiution !)} \end{aligned}$$

now the C.F. is $= A + Be^{-2x}$
thus the G.S. is $= A + Be^{-2x} + x^3/6 - x^2/4 + x/4$ ✓

This method of expanding $1\ /\ ((D-A)(D-B))$ always works if $f(x)$ is a polynomial.

1.10.4 Finding the Particular Integral if $f(x)$ is Not a Polynomial

Here there is no universal method, but some equations are solvable.

E.g. consider $f(x) = e^{cx}\cdot \Phi(x)$ $\quad$ where Φ is a polynomial
thus

$$\begin{aligned} Df &= e^{cx}(D\Phi + c\Phi) \\ &= e^{cx}(D+c)\,\Phi \end{aligned}$$

therefore $D^2 f = D^2 e^{cx}\Phi = e^{cx}(D+c)^2\,\Phi$
moreover $(D-a)f = e^{cx}(D+c-a)\,\Phi$
in particular $(D-c)f = e^{cx}\,D\,\Phi$
and moreover $(D-a)^{-1}f = e^{cx}(D+c-a)^{-1}\,\Phi$

[proof :

$$\begin{aligned} (D-a)(D-a)^{-1}f &= (D-a)\,e^{cx}(D+c-a)^{-1}\,\Phi \\ &= e^{cx}(D+c-a)(D+c-a)^{-1}\,\Phi \\ &= e^{cx}\,\Phi = f \end{aligned}$$

Q.E.D.]

so our P.I. is $y = [1\,/\,((D-a)(D-b))]\,e^{cx}\,\Phi$
reduces to $y = e^{cx}\,[1\,/\,((D+c-a)(D+c-b))\,]\,\Phi$

which can be evaluated as in the previous section, because Φ is a polynomial.

e.g. $(D^2 + D - 2)\, y = x\, e^x$

or $y = [1\,/\,((D+2)(D-1))]\, x\, e^x$

so the P.I. is
$$= e^x\,[1\,/\,((D+3)D)]\, x$$
$$= (e^x/3)\,(1 - D/3 + D^2/9 - \ldots)\, D^{-1}\, x$$
$$= (e^x/3)\,(1 - D/3 + D^2/9 - \ldots)\, x^2/2$$
$$= (e^x/3)\,(x^2/2 - x/3 + 1/9 - 0 + 0 \ldots)$$

thus the G.S. is $y = Ae^{-2x} + Be^x + (e^x/3)\,(x^2/2 - x/3 + 1/9)$ ✓

By taking real parts, this approach works if instead of e^x we have a cosine (or e^x times a cosine). Moreover, it still works if instead of x we have a polynomial $\Phi(x)$.

e.g. $f(x) = \Phi(x) \cdot cos(cx)$ where Φ is a polynomial

consider $(D^2 + AD + B)\, y = \Phi \cdot cos(cx)$

i.e. $y = [1\,/\,((D-a)(D-b))] \cdot \Phi \cdot cos(cx)$

now $cos(cx) = \Re\,(e^{icx})$

thus
$$y = \Re\,[1\,/\,((D-a)(D-b))]\, \Phi\, e^{icx}$$
$$= \Re\, e^{icx}\,[1\,/\,((D+ic-a)(D+ic-b))]\, \Phi$$

which can be evaluated as in the pre-previous section if Φ is a polynomial, or as in the previous section if it is e^x times a polynomial.

e.g. $(D-1)^2\, y = cos\, 3x$

i.e.
$$y = [1/(D-1)^2]\, cos\, 3x$$
$$= \Re\,[1/(D-1)^2]\, e^{i3x}$$
$$= \Re\, e^{3ix}\,[1/(D+3i-1)^2] \cdot 1$$
$$= \Re\, e^{3ix} \cdot 1/(3i-1)^2$$
$$= \Re\, e^{3ix} \cdot (3i+1)^2/(-9-1)^2$$
$$= \Re\, e^{3ix} \cdot (-9+1+6i)/100$$
$$= (-8\, cos\, 3x + i \cdot i \cdot 6 sin\, x)/100$$

thus the G.S. is $y = (A + Bx)\, e^x - (8\, cos\, 3x + 6\, sin\, x)/100$ ✓

Similarly, by taking imaginary parts we can solve $f(x) = \Phi(x) \cdot sin(cx)$ ...

e.g. $(D^2 + 4)\, y = x\, sin\, 2x$

i.e.
$$y = (1\,/\,(D^2+4)) \cdot x\, sin\, 2x$$
$$= \Im\,(1\,/\,(D^2+4)) \cdot x\, e^{2ix}$$
$$= \Im\, e^{2ix}\,(1\,/\,((D+2i)^2+4)) \cdot x$$
$$= \Im\, e^{2ix}\,(1\,/\,(D^2+4iD)) \cdot x$$
$$= \Im\, e^{2ix}\,(1\,/\,(4iD))\,(1 - D/4i + D^2/(4i)^2 - \ldots) \cdot x$$
$$= \Im\, e^{2ix}\,(1\,/\,(4iD))\,(x + i/4)$$
$$= \Im\, e^{2ix}\,(x/16 - ix^2/8)$$
$$= -cos\, 2x \cdot x^2/8 + sin\, 2x \cdot x/16$$

thus the G.S. is
$$y = Ae^{2ix} + Be^{-2ix} - x^2/8 \cdot cos\, 2x + x/16 \cdot sin\, 2x$$
$$= a\, sin\, 2x + b\, cos\, 2x - x^2/8 \cdot cos\, 2x + x/16 \cdot sin\, 2x$$ ✓

1.11 Linear D.E. with constant coefficients

This is a D.E with the form

$$d^n y/dx^n + a_1 d^{n-1}/dx^{n-1} + \cdots + a_n y = f(x)$$

and as with the 2nd-order case (Section 1.10) ,

$$general\ solution = complimentary\ function + particular\ integral$$

1.11.1 Finding the Complimentary Function

This is analagous to the 2nd-order case. The C.F. is the solution of

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n\right) y = 0$$

and the auxiliary equation is of degree n

$$(m - \alpha_1)(m - \alpha_2)\ldots(m - \alpha_n) = 0$$

This has n roots: $\quad m = \alpha_1, \alpha_2, \ldots \alpha_n \quad$ If these roots are all distinct,

$$C.F. = y = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} + \cdots + A_n e^{\alpha_n x}$$

and if s of the α's are the same, e.g. $\quad \alpha_1 = \alpha_2 = \cdots = \alpha_s \quad$ then:

$$C.F. = y = \left(A_1 + A_2 x + A_3 x^2 + \cdots + A_s x^{s-1}\right) e^{\alpha_1 x} + A_{s+1} e^{\alpha_{s+1} x} + \cdots + A_n e^{\alpha_n x}$$

1.11.2 Finding the Particular Integral

$$P.I. = y = \left[1 \,/\, \left(D^n + a_1 D^{n-1} + \cdots + a_n\right)\right)] \cdot f(x)$$

is tackled in a precisely analagous manner to the 2nd-order case of Sections 1.10.3 to 1.10.5

1.12 Homogeneous Linear D.E.

This is a D.E. of the form

$$x^n (d^n y/dx^n) + a_1 x^{n-1} (d^{n-1} y/dx^{n-1}) + \cdots + a_n y = f(x)$$

This can be solved with the substitution $\quad x = e^t$

$$\begin{aligned}
&\text{let} & x &= e^t \\
&\text{so that} & dy/dx &= (dy/dt) \,/\, (dx/dt) = (dy/dt)/x \\
&\text{i.e.} & D &= (1/x) \cdot d/dt \\
&\text{and} & D^2 y &= (1/x) \cdot d(Dy)/dt \\
& & &= \frac{1}{x} \cdot \left(\frac{-1}{x^2} \frac{dx}{dt} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \right) \\
& & &= (1/x^2) \cdot (-dy/dt \,+\, d^2 y/dt^2) & &\text{since } dx/dt = x \\
&\text{further} & D^3 y &= (1/x) \cdot d/dt \cdot \left[(1/x^2)\left(-dy/dt + d^2 y/dt^2\right)\right] \\
& & &= \frac{1}{x} \left(\frac{-2}{x^3} \frac{dx}{dt} \left(-\frac{dy}{dt} + \frac{d^2 y}{dx^2}\right) + \frac{1}{x^2} \left(-\frac{d^2 y}{dx^2} + \frac{d^3}{dt^3}\right) \right) \\
& & &= \frac{1}{x^3} \left(2\frac{dy}{dt} - 3\frac{d^2 y}{dx^2} + \frac{d^3 y}{dt^3} \right)
\end{aligned}$$

The faxtor $1/x^r$ at the beginning of these expressions will cancel with the x^r in the D.E. and will reduce it to a Linear D.E. with Constant Coefficients.

e.g. $$(x^3D^3 + 3x^2D^2 + xD)\,y = 24x^2$$
let $$x = e^t$$
so $$D = (1/x)\cdot d/dt$$
$$D^2 = (1/x^2)\cdot(-d/dt + d^2/dt^2)$$
$$D^3 = (1/x^3)(2d/dt - 3d^2/dt^2 + d^3/dt^3)$$
i.e. $$2dy/dt - 3d^2y/dt^2 + d^3y/dt^3 - 3dy/dt + 3d^2y/dt^2 + dy/dt = 24e^{2t}$$
i.e. $$d^3y/dt^3 = 24e^{2t}$$
$$d^2y/dt^2 = 12e^{2t} + A'$$
$$dy/dt = 6e^{2t} + A't + B$$
$$y = 3e^{2t} + At^2 + Bt + C$$
or $$y = 3x^2 + A(log\,x)^2 + B\,log\,x + C \quad \checkmark$$

1.13 Simultaneous Linear D.E.'s with Constant Coefficients

e.g. $$dx/dt + dy/dt - 3x - 15y = -4t \tag{12}$$
and $$dx/dt + 2dy/dt + x = -5t^2 \tag{13}$$
So we will redefine D $$D \equiv d/dt$$
from (12) $$(D-3)x + (D-15)y = -4t \tag{14}$$
from (13) $$(D+1)x + 2Dy = 5t^2 \tag{15}$$
(14) x 2D $$2D(D-3)x + 2D(D-15)y = -8 \tag{16}$$
(15) x (D-15) $$(D-15)(D+1)x + 2D(D-15)y = 10t - 75t^2 \tag{17}$$
(16) - (17) $$(2D^2 - 6D - D^2 - 14D - 15))x = 75t^2 - 10t - 8$$
or $$(D^2 + 8D + 15)x = 75t^2 - 10t - 8$$
or $$(D+3)(D+5)x = 75t^2 - 10t - 8$$

so the C.F. is $x = Ae^{-3t} + Be{-}5t$

and the P.I. is
$$\begin{aligned} x &= [1\,/\,((D+3)(D+5))]\,(75t^2 - 10t - 8) \\ &= [1\,/\,((1+D/3)(1+D/5))]\,(5t^2 - 2t/3 - 8/15) \\ &= [1 - D/3 + D^2/9 - \ldots]\,[1 - D/5 + D^2/25 - \ldots]\,(5t^2 - 2t/3 - 8/15) \\ &= [1 - D/3 + D^2/9 - \ldots]\,[5t^2 - 2t/3 - 8/15\ - 2t + 2/15\ + 6/15] \\ &= [1 - D/3 + D^2/9 - \ldots]\,[5t^2 - 8t/3] \\ &= 5t^2 - 8t/3 - 10t/3 + 8/9 + 10/9\ =\ 5t^2 - 6t/3 + 2 \end{aligned}$$

thus G.S. is $x = Ae^{-3t} + Be^{-5t} + 5t^2 - 6t + 2$ ✓

We find y by eliminating dy/dt between (12) & (13), as this gives a simple algebraic equation in y

2x(12) - (13) $$dx/dt - 7x - 30y = -8t - 5t^2$$
i.e. $$y = (1/30)\,(-10Ae^{-3t} - 12Be^{-5t} - 20 + 60t - 30t^2) \quad \checkmark$$

or e.g. $$Dy + x = 0 \tag{18}$$
and $$Dx - y = 0 \tag{19}$$
D x (19) $$D^2x - Dy = 0 \tag{20}$$
(18) + (20) $$D^2x + x = 0$$
thus $$x = Acos\,t + Bsin\,t \quad \checkmark$$ as in Section 1.2
subst in (19) $$y = -Asin\,t + Bcos\,t \quad \checkmark$$

1.14 Exact D.E. of the 1st order

Consider a function $\Phi(x,y)$ Then $d\Phi = (\delta\Phi/\delta x)dx + (\delta\Phi/\delta y)dy$
This is a total, or exact, differential.

Note that, assuming differentiability, $\delta^2\Phi/\delta \cdot \delta y = \delta^2\Phi/\delta y\delta x$
Suppose we now have a 1st-order D.E. $dy/dx = f(x,y)$
Rearranging , $M(x,y)dx + N(x,y)dy = 0$ (21)

If now M and N happen to be such that (21) can be expressed as $d\Phi = 0$ then the solution of the D.E. is, integrating, $\Phi(x,y) = 0$

e.g. $xdy + ydx = 0$
can be expressed as $d(xy) = 0$
integrating $xy = c$ ✓

1.14.1 Necessary condition on M and N for $M(x,y)dx + N(x,y)dy$ to be exact

If $\Phi(x,y)$ exists, and $D\Phi(x,y) = Mdx + Ndy$
then $M \equiv \delta\Phi/\delta x$ and $N \equiv \delta\Phi/\delta y$
so, by differentiabilty, $\delta M/\delta y = \delta N/\delta x$ (22)

1.14.2 Sufficient condition on M and N for $M(x,y)dx + N(x,y)dy$ to be exact

Is that also a sufficient condition ? i.e., given $\delta M/\delta y = \delta N/\delta x$ what becomes of (21) ?
Let us define $u(x,y)$ by $M = \delta u/\delta x$ where u is differentiable.

thus $\delta M/\delta y = \delta^2 u/\delta y\delta x$
thus from (22) $\delta N/\delta x = \delta^2 u/\delta y\delta x$
integrating $\delta u/\delta y = N + f(y)$ where f is an arbitrary function
subst in (21) $(\delta u/\delta x)dx + (\delta u/\delta y - f(y))dy = 0$ (23)
but $du = (\delta u/\delta x)dx + (\delta u/\delta y)dy$ (24)
thus (24) in (23) $du - f(y)dy = 0$

and since we can choose $f \equiv 0$ this is of the form $d\Phi = 0$ Q.E.D.

e.g. $(x - y + z)dx - (x + y - 1)dy = 0$
now since $\delta M/\delta y = -1 = \delta N/\delta x$ this is exact
and $\delta\Phi/\delta x = x - y + 2$ thus $\Phi = x^2/2 - xy + 2x + f(y)$
and $\delta\Phi/\delta y = -x - y + 1$ thus $\Phi = -xy - y^2/2 + y + g(x)$ so we identify f and g
therefore $\Phi = x^2/2 - y^2/2 - xy + 2x + y$ plus a constant
the solution is $\Phi = \text{const.}$
i.e. $x^2/2 - y^2/2 - xy + 2x + y = c$ ✓

e.g. $M(x)dx - N(y)dy = 0$ see Section 1.5

gives $$\Phi = \int M(x)dx + \int N(y)dy = c$$

so this method also includes the Separable D.E. of the 1st order as a special case.

1.15 Reduction to Exact Form by an Integrating Factor

If	$Mdx + Ndy \neq 0$	is not exact,	
nevertheless	$\mu Mdx + \mu Ndy = 0$	may still be exact;	
i.e.	$\delta(\mu M)/\delta y = \delta(\mu N)/\delta x$		(25)

Then μ is called an 'integrating factor'. (compare Section 1.8)

e.g.	$(y/x)dx + dy = 0$	is not exact
but multiply by x,	$ydx + xdy = 0$	and this is exact
	$d(xy) = 0$	
	$x = c$	✓

$\mu(x, y)$ is found from (25)	$\delta(\mu M)/\delta y = \delta(\mu N)/\delta x$
or	$M(\delta\mu/\delta y) + \mu(\delta M/\delta y) = N(\delta\mu/\delta x) + \mu(\delta N/\delta x)$

This is intractable. However, there are certain D.E.'s for which it becomes tractable ; for example, there are D.E.'s which give $\mu = \mu(x)$ only, not $\mu = \mu(x, y)$

so that	$\delta\mu/\delta y = 0$
and	$\delta\mu/\delta x = d\mu/dx$
thus (23) simplifies to	$\mu\,(\delta M/\delta y) = N\,(d\mu/dx) + \mu\,(dN/dx)$
i.e.	$N\,(d\mu/dx) = \mu\,(\delta M/\delta y\ -\ \delta N/\delta x)$
i.e.	$(1/\mu)\,(d\mu/dx) = (1/N)\,(\delta M/\delta y\ -\ \delta N/\delta x)$

Now for $\mu = \mu(x)$, the LHS is a function of x, thus if we are given M and N such that $(1/N)\,(\delta M/\delta y\ -\ \delta N/\delta x) = func(x)$ we will probably be able to integrate (24) to find a μ.

1.16 Linear D.E.'s of the 1st order revisited

Linear D.E.'s of the 1st order (see Section 1.8) represent a particular application of this technique.

for if	$dy/dx + P(x)\,y = Q(x)$	
although	$(Py - Q)dx + dy = 0$	is not exact,
we have	$1/N\,(\delta M/\delta y - \delta N/\delta x) = P = func(x)$	satisfying (25)
thus (24) is	$1/\gamma\,(\delta\mu/dx) = P(x)$	
i.e.	$log\,\mu = \int Pdx$	
i.e.	$\mu = exp(\int Pdx)$	

which derives the integrating factor that we introduced arbitrarily in Section 1.8.

multiplying,	$(Py - Q)\,exp(\int Pdx)\,dx + exp(\int Pdx)\,dy = 0$	
should be exact; i.e.	$\delta\Phi/\delta x = (Py - Q)\,exp(\int Pdx)$	(26)
and	$\delta\Phi/\delta y = exp(\int Pdx)$	(27)
integrating (26)	$\Phi = y\,exp(\int Pdx) - \int Q\,exp(\int Pdx)dy + A(y)$	
integrating (27)	$\Phi = y\,exp(\int Pdx) + B(x)$	
combining,	$\Phi = y\,exp(\int Pdx) - \int Q\ exp(\int Pdx)\,dy = C$ ✓	

1.17 First Order D.E.'s with One Variable Absent

If we put $p \equiv dy/dx$ then the general 1st order D.E. is $f(x, y, p) = 0$
If p is absent, then $f(x, y) = 0$ is the solution.
If x is absent, then $f(y, p) = 0$ can often be reduced to one of two simple cases:

1) If p is a function of y $\quad p = \Phi(y)$

i.e. $\quad dx = dy \,/\, \Phi(y)$ $\quad$ which is separable

2) If y is a function of p $\quad y = \Phi(p)$

differentiating $\quad p = dy/dx = \Psi'(p) \cdot (dp/dx)$ $\quad$ which is also separable

i.e. $\quad x = \int (\Psi'(p) \,/\, p)\, dp + c$

and $\quad y = \Psi(p)$ $\quad$ with a parametric solution $x(p)$ and $y(p)$

If y is absent, then $\quad f(x,p) = 0 \quad$ and we distinguish the same two simple cases:

1) If p is a function of x $\quad p = \Phi(x)$

$$dy = \Phi(x)\, dx$$

i.e. $\quad y = \int \Phi(x)\, dx$

2) If x is a function of p $\quad x = \Phi(p)$

differentiating $\quad 1 = \Psi'(p) \cdot (dp/dx)$

$$1 = \Psi'(p) \cdot p \cdot (dp/dy)$$

integrating $\quad y = \int p\Psi'(p) dp \ + c$

and $\quad x = \Psi(p)$ $\quad$ with a parametric solution $x(p)$ and $y(p)$

E.g. $\quad p^2 + 2yp = 3y^2$

$$(p + 3y)(p - y) = 0$$

$$p = -3y \quad \text{or} \quad p = y$$

$\therefore$ $\quad log\, y = -3x + A$

or $\quad log\, y = x + B$

i.e. $\quad y = c_1 exp(-3x)$

or $\quad y = c_2 exp(x)$ $\quad$ ✓

E.g. $\quad 3p^5 - py + 1 = 0$

i.e. $\quad y = 3p^4 + 1/p$

differentiating $\quad p = (12p^3 - 1/p^2)\,(dp/dx)$

separating $\quad x = \int (12p^2 - 1/p^3)\, dp + c \ = \ 4p^3 + 1/2p^2 + c$

and $\quad y = 3p^4 + 1/p$ $\quad$ ✓

E.g. $\quad x = p + p^4$

differentiating $\quad 1 = (1 + 4p^3)\,(dp/dx) \ = \ (p + 4p^4)\,(dp/dy)$

separating $\quad y = \int (p + 4p^4)\, dp \ = \ p^2/2 + 4p^5/5 + c$

and $\quad x = p + p^4$ $\quad$ ✓

1.18 Clairault's Equation

This is an equation of the form $\quad y = x \cdot p + f(p) \quad$ where $\quad p \equiv dy/dx$

differentiating, $\quad p = p + x(dp/dx) + f'(p)(dp/dx)$

or $\quad (x + f'(p))\,(dp/dx) = 0$

therefore either $\quad dp/dx = 0$

or $\quad x + f'(p) = 0$

1) If $dp/dx = 0$ then	$p = c$	
substituting,	$y = cx + f(c)$	✓ the General Solution
2) If $x + f'(p) = 0$ then	$x = -f'(p)$	
substituting,	$y = -pf'(p) + f(p)$	✓ a Singular Solution

This solution contains no arbitrary constants, but it cannot be found from the G.S. !
It is called a 'singular solution'.

e.g.	$y = xp + 1/p$	
differentiating,	$p = p + x(dp/dx) - (1/p^2)\, dp/dx$	
	$(x - 1/p^2)\,(dp/dx) = 0$	
thus either	$(dp/dx) = 0$ thus $p = c$	
substituting	$y = cx + 1/c$	✓ the General Solution
or	$x = 1/p^2$	
substituting	$y = 2/p$	
separating,	$y^2 = 4x$	✓ a Singular Solution

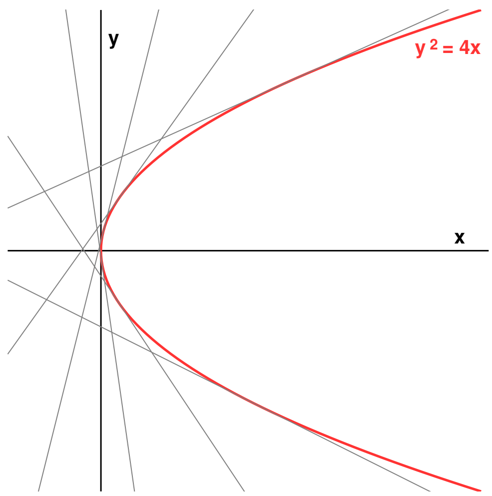


Figure 1: The Singular Solution $y^2 = 4x$ as the Envelope of the family $y = cx + 1/c$

The curve $y^2 = 4x$ is the envelope of the family of lines $y = cx + 1/c$
and since they are tangential, dy/dx is the same.

1.19 Second order D.E.'s with one variable absent

The general second-order D.E. is $F(x,\, y,\, y',\, y'') = 0$

1.19.1 If y is absent

If y is absent, then we have $F(x,\, y',\, y'') = 0$
This is easy: we put $y' = p$, and get a 1st-order equation in p and x : $F(x,\, p,\, p') = 0$

e.g.	$x\, y'' = 1 + y'$	
i.e.	$x\, dp/dx = 1 + p$	
i.e.	$\int dp/(1+p) = \int dx/x + A$	
i.e.	$log\,(1+p) = log\, x + A$	
i.e.	$p = Bx - 1$	
i.e.	$y = Cx^2 - x + D$	✓

1.19.2 If x is absent

If x is absent, then we have $f(y, y', y'') = 0$ So $y'' = dp/dx = p \cdot dp/dy$
which gives us $g(y, p, dp/dy) = 0$ which is a 1st-order equation in p and y.

e.g.	$y'' = 2y^3/a^4$	
given the B.C.	$y = a$ and $y' = 1$ when $x = 0$	
i.e.	$y'' = p \cdot dp/dy = 2y^3/a^4$	
separating	$p^2 = (y/a)^4 + A$	
but from $y = a$ when $p = 1$	$A = 0$	
i.e.	$p^2 = (y/a)^4$	
i.e.	$p = (y/a)^2$	(+ve because of the B.C.)
separating	$-a^2/y = x + B$	
but from $y = a$ when $x = 0$	$B = -a$	
thus	$-a^2/y = x - a$	
or	$y = a^2/(a - x)$	✓

1.20 General Linear D.E. of the 2nd Order

The general linear second-order D.E. is	$A(x)y'' + B(x)y' + C(x)y = E(x)$	(28)
its Complimentary Function is the G.S. of	$A(x)y'' + B(x)y' + C(x)y = 0$	(29)
and its Particular Integral is any solution of	$A(x)y'' + B(x)y' + C(x)y = E(x)$	

We can reduce (28) to 1st order if we happen to know a $z(x)$ which is a P.I. of (28),
or even a P.I. of (29).

To do this, we let	$y = w \cdot z$
where z is known to satisfy	$Az'' + Bz' + C = 0$
so that	$y' = w'z + wz'$ and $y'' = w''z + 2w'z' + wz''$
substituting	$A\left(w''z + 2w'z' + wz''\right) + B\left(w'z + wz'\right) + C(z) = E$
using $Az'' + Bz' + C = 0$:	$Azw'' + \left(2Az' + Bz\right)w' = E$

and since $z(x)$ is known, we now have a 1st-order equation in w'.

e.g.	$(2x + x^2)y'' - 2(1 + x)y' + 2y = 0$	
given that	$y = x^2$	is a P.I.
We let	$y = w(x) \cdot x^2$	be the G.S.
so that	$(2x + x^2)(w''x^2 + 4xw' + 2w) \; - \; 2(1 - x)(w'x^2 + 2xw) \; + \; 2wx^2 = 0$	
i.e.	$(2x^3) + x^4)w'' \; + \; (8x^2 + 4x^3 - 2x^2 - 2x^3)w' = 0$	
i.e.	$w''/w' = (6 + 2x)/(x + 2x^2) = (-3/x) + 1/(2 + x)$	
integrating	$log\, w' = -3 log\, x + log(2 + x) + a$	
antilogs,	$w' = b(2 + x)/x^3$	
integrating	$w = -b/x^2 - b/x + c$	
thus G.S is	$y = wx^2 = -b(1 + x) + cx^2$	✓

1.21 Exact Linear D.E. of the 2nd Order

If, in the equation $A(x)y'' + B(x)y' + C(x)y = E(x)$
the L.H.S is $= d\left(f(x, y, y')\right)/dx$ then the equation is said to be 'exact',
and its 'First Integral' is the 1st-order D.E. $f(x, y, y') + \int E dx \; + \; c$

e.g. $$x^2y'' + xy' - y = x^2$$

integrating term by term, $\int x^2y''dx = x^2y' - \int 2xy'dx = x^2y' - 2xy + \int 2ydx$

2nd term by parts $\int xy'dx = xy - \int ydx$

3rd term $\int -ydx = -\int ydx$

summing $x^2y' - xy + \int(2y - y - y)dx = x^2y' - xy = x^3/3 + a$ ✓

In a similar manner, we can derive a **condition for exactness**.

Consider $$Ay'' + By' + Cy = E$$

term by term: $Ay'' = Ay' - \int A'y' = Ay' - a'y + \int A''ydx$

2nd term $\int By' = By - \int B'y$

3rd term $\int Cy = \int Cy$

summing, the D.E. is $Ay' - Ay' + By + \int(A'' - B' + C)y = E + a$

which is exact if $A'' - B' + C = 0$ and the 'first integral' is $Ay' + (-A' + B)\,y$

1.22 Reduction to Exact Form by an Integrating Factor

If $Ay'' + By' + Cy = E$ is not exact,

i.e. $A'' - B' + C \neq 0$

nevertheless $\mu Ay'' + \mu By' + \mu Cy = \mu E$ may still be exact

i.e. $(\mu A)'' - (\mu B)' + \mu C = 0$

or $d(\mu A)/dx - d(\mu B)/dx + \mu C = 0$ then $\mu(x)$ is called an 'integrating factor'

E.g.: Show that e^x is an I.F. of $y'' + xy' + xy$
and hence solve $y'' + xy' + xy = 0$ subject to $y' = 0$ when $x = 1$.

To show that $e^xy'' + xe^xy' + xe^xy = 0$ is exact,

we evaluate $d^2(e^x)/dx^2 - d(xe^x)/dx + xe^x$

$= e^x - xe^x - e^x + xe^x = 0$ therefore it is exact.

The 1st integral is $e^xy' - e^xy + xe^xy = a$

substituting $x = 1, y' = 0$ $\quad -e^xy + e^xy = a$

so $a = 0$

substituting $y' - y + xy = 0$

separating $dy/y = (1 - x)dx$

integrating $log\, y = x - x^2/2 + b$ ✓

1.23 Solution in Series (Frobenius' Method)

This is a series method of solving $P(x)y'' + Q(x)y' + R(x) = 0$
which assumes that $y = x^t\,(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$

so that
$$y = \sum_0^\infty a_n\, x^{n+t}$$
$$y' = \sum_0^\infty (n+t)\, a_n\, x^{n+t-1}$$
$$y'' = \sum_0^\infty (n+t)(n+t-1)\, a_n\, x^{n+t-2}$$

E.g. $$2xy'' + y' - xy = 0$$

substituting $$\sum_0^\infty \left(2(n+t-1)+(n+t)\right) a_n x^{n+t-1} + \sum_0^\infty a_n x^{n+t-1} \equiv 0$$

Now this is an identity for all x ; therefore all coefficients on the L.H.S must vanish.

the coefficient of x^{t-1} is $[2(t-1)t + t]a_0 = 0$

assuming $a_0 \neq 0$, $2(t-1)t = 0$

i.e. $t = 0$ or $t = 1/2$

the coefficient of x^t is $[2t(t+1) + (1+t)]\, a_1 = 0$

i.e. $(1+t)\,(2t+1)\, a_1 = 0$

and since $t = 0$ or $t = 1/2$, $a_1 = 0$

the coefficient of x^{n+t-1} is $(n+t)\,(2n+2t-1)\, a_n - a_{n-2} = 0$

i.e. $$a_n = a_{n-2}/[(n+t)(2n+2t-1)] \qquad (30)$$

thus for n odd, $\cdots = a_7 = a_5 = a_3 = a_1 = 0$

whereas for n even we have two possible solutions :

1) $t = 0$, giving $a_n = a_{n-2} \,/\, [n(2n-1)]$

thus $a_2 = a_0/(2\times 3)$ and $a_4 = a_0/(2\times 4\times 3\times 7)$

and $a_6 = a_0/(2\times 4\times 6\times 3\times 7\times 11)$

thus $y = a_0(1 + x^2/(2\times 3) + x^4/(2\times 4\times 3\times 7) + x^6/(2\times 4\times 6\times 3\times 7\times 11) + \cdots)$

2) $t = 1/2$, giving $y = a_0 x^{1/2}(1 + x^2/(2\times 5) + x^4/(2\times 4\times 5\times 9) + \cdots)$

Therefore the following is a General Solution :

$$y = A(1 + x^2/(2\times 3) + x^4/(2\times 4\times 3\times 7)\cdots) + Bx^{1/2}((1 + x^2/(2\times 5) + x^4/(2\times 4\times 5\times 9)\cdots)$$

On the Convergence of Series . . .

If $$s = \sum_0^\infty u_n$$

then if $\lim_{n\to\infty} |u_{n+1}/u_n| < 1$ then the series converges

whereas if $\lim_{n\to\infty} |u_{n+1}/u_n| > 1$ then the series diverges.

E.g. for (30) $$\lim_{n\to\infty} |\frac{a_n}{a_{n-2}} x^2| = \lim_{n\to\infty} |x^2/[(n+t)(2n+2t-1)]| = 0$$

So the series (30) is convergent for both values of t, and for all x.

Or e.g. $$y'' - y = 0$$

substitute as before $$y = \sum_0^\infty a_n x^{n+t} \quad \text{so that} \quad y'' = \sum_0^\infty (n+t)(n+t-1) a_n x^{n+t-2}$$

we get $$\sum_0^\infty (n+t)(n+t-1) a_n x^{n+t-2} - \sum_0^\infty a_n x^{n+t} = 0$$

the coefficient of x^{t-2} $$(t-1)t\, a_0 = 0 \qquad (31)$$

coefficient of x^{t-1} $$t(t+1)\, a_1 = 0 \qquad (32)$$

coefficient of x^{t+n-2} $$(n+t-1)(n+t)a_n - a_{n-2} = 0 \qquad (33)$$

from (31) and (32), $$\text{if } a_0 \neq 0 \text{ then } t = 0 \text{ or } 1 \text{ so } a_1 = 0 \qquad (34)$$

likewise $$\text{if } a_1 \neq 0 \text{ then } t = 0 \text{ or } -1 \text{ so } a_0 = 0 \qquad (35)$$

or $$\text{if } a_1 \neq 0 \text{ and } a_1 \neq 0 \text{ then } t = 0 \qquad (36)$$

Each of these solutions will give us an equivalent G.S. We will take solution (36).

Putting $t = 0$ in (33) $\quad a_n = a_{n-2}/((n-1)n)$

i.e. $\quad a_2 = a_0/2!\ ,\ a_3 = a_1/3!\ ,\ a_4 = a_0/4!\ ,\ a_5 = a_1/5!\ ,$ etc

thus
$$y = a_0(1 + x^2/2! + x^4/4! + \cdots) + a_1(x + x^3/3! + x^5/5! + \cdots)$$
$$= a_0 \cosh x + a_1 \sinh x \qquad \checkmark$$

We will show that taking solution (34) gives an equivalent solution :

Putting $t = 1$ in (33) $\quad a_n = a_{n-2}/(n(n+1))$

i.e. $\quad a_2 = a_0/3!\ ,\ a_3 = 0/4!\ ,\ a_4 = a_0/5!\ ,\ a_5 = 0\ ,$ etc

thus
$$y = x_0\,(a_0 + a_1x + a_2x^2 + \cdots)$$
$$= a_0\,(x + x_3/3! + x^5/5! + \cdots)$$
$$= a_0 \sinh x$$

And similarly, the case $t = 0$ gives us $\quad y = a_1 \cosh x \qquad \checkmark$

1.24 Legendre's Equation

Legendre's Equation is $\quad (1 - x^2)y'' - 2xy' + l(l+1)y = 0 \quad$ where l is known as the 'order'. It can be solved by Frobenius' Method.

let
$$y = \sum_0^\infty a_n x^{n+t}$$

so that
$$y' = \sum_0^\infty (n+t)a_n x^{n+t-1}$$

and
$$y'' = \sum_0^\infty (n+t)(n+t-1)a_n x^{n+t-2}$$

Substituting in Legendre's Equation,

$$\sum (n+t)(n+t-1)a_n x^{n+t-2} - \sum (n+t)(n+t-1)a_n x^{n+t}$$
$$-2\sum (n+t)a_n x^{n+t} - \sum l(l+1)a_n x^{n+t} = 0$$

i.e.
$$\sum (n+t)(n+t-1)a_n x^{n+t-2} - \sum [(n+t-1)(n+t) - l(l+1)]\, a_n x^{n+t} = 0$$

The lowest power of x is x^{t-2}, and its coefficient is $\qquad (t-1)\,t\,a_0 = 0 \qquad (31)$

the coefficient of x^{t-1} gives $\qquad t\,(t+1)\,a_1 = 0 \qquad (32)$

As in the previous section, we will take $t = 0,\ a_0 \neq 0,\ a_1 \neq 0$ to get the General Solution.

coefft of x^{n+t-2} $\quad (n+t-1)(n+t)a_n - [(n+t-1)(n+t-2) - l(l+1)]a_{n-2} = 0$

setting $t = 0$ $\quad a_n = a_{n-2}[(n-1)(n-2) - l(l+1)]\,/\,(-1)n$

so for $n = 2$ $\quad a_2 = -l(l+1)a_0\,/\,2!$

for $n = 3$ $\quad a_3 = [2.1 - l(l+1)]\,a_1\,/\,3!$

for $n = 4$ $\quad a_4 = [3.2 - l(l+1)]\,a_2\,/\,4.3 = -[3.2 - l(l+1)]\,l(l+1)\,a_0$

for $n = 5$ $\quad a_5 = [4.3 - l(l+1)]\,[2.1 - l(l+1)]\,a_1\,/\,5!$

thus the G.S. is
$$y = a_0\,[1 - l(l+1)x^2/2! - (3.2 - l(l+1))\,l(l+1)x^4/4! \cdots] +$$
$$+ a_1\,[x + (2.1 - l(l+1))x^3/3! + (4.3 - l(l+1))(2.1 - l(l+1))x^5/5! + \cdots] \quad \text{etc}$$

If l is not a positive integer, both series are infinite in length.

1.25 Legendre Polynomials

Legendre Polynomials, also known as Zonal Harmonics, arise when l is a positive integer.

If $l = 0$ the second series remains infinite, but the first becomes $= a_0$
If $l = 1$ the first series remains infinite, but the second becomes $= a_1 x$ (since $2.1 - l(l+1) = 0$)
If $l = 2$ the second series remains infinite, but the first becomes $= a_0(1 - 3x^2)$ (since $3.2 - l(l+1) = 0$)
If $l = 3$ the first series remains infinite, but the second becomes $= a_0(x - 5x^3/3)$
These are called the Legendre Polynomials, or Zonal Harmonics.
The Legendre Polynomial of order l is denoted by $P_l(x)$ The first few are:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= 1/2 \cdot (3x^2 - 1)\\
P_3(x) &= 1/2 \cdot (5x^3 - 3x)\\
P_4(x) &= 1/8 \cdot (35x^4 - 30x^2 + 3)\\
P_5(x) &= 1/8 \cdot (63x^5 - 70x^3 + 15x)
\end{aligned}$$

where a_0 and a_1 have been chosen according to a convention.
We can see that the highest-order term is x^l, which is why l is known as the 'order'.

$$\begin{aligned}
\text{from (33)} \quad a_{n-2} &= ([n(n-1)] \,/\, [n-2)(n-1) - n(n-1)]) \cdot a_n\\
&= n(n-1)a_n \,/\, (n^2 - 3n + 2 - n^2 - n)\\
&= -n(n-1)a_n \,/\, (2(2n-1))\\
\text{also,} \quad a_{n-4} &= -(n-2)(n-3)a_{n-2} \,/\, (4(2n-3))\\
&= n(n-1)(n-2)(n-3)a_n \,/\, (2 \cdot 4(2n-1)(2n-3))\\
\text{thus} \quad P_n(x) &= a_n[x_n - n(n-1)a_n/2(2n-1) + n(n-1)(n-2)(n-3)a_n/2 \cdot 4(2n-1)(2n-3)]
\end{aligned}$$

a_n is conventionally given the arbitrary value $\quad a_n = (2_n - 1)(2n-3)\ldots 3 \cdot 1 \,/\, n \quad$ for $n \neq 0$
and $\quad a_n = 1 \quad$ for $n = 0$

1.26 Bessel's Equation

Bessel's Equation is $x^2y'' + xy' + (x^2 - l^2)y = 0$ where l is known as the 'order'.
It can be solved by Frobenius' Method. All sums are from 0 to ∞.

$$\begin{aligned}
\text{let} \quad y &= \sum a_n x^{n+t}\\
\text{so that} \quad y' &= \sum (n+t)a_n x^{n+t-1}\\
\text{and} \quad y'' &= \sum (n+t)(n+t-1)a_n x^{n+t-2}
\end{aligned}$$

$$\text{Substituting,} \quad \sum (n+t)(n+t-1)a_n x^{n+t} + \sum (n+t)a_n x^{n+t} + \sum a_n x^{n+t+2} - \sum l^2 a_n x^{n+t} = 0$$

$$\text{i.e.} \quad \sum [(n+t)^2 - l^2]a_n x^{n+t} + \sum a_n x^{n+t+2} = 0$$

The lowest power of x is x^t ; its coefficient is $\quad (t^2 - l^2)\, a_0 = 0$
and x^{t+1} gives $\quad ((t+1)^2 - l^2)\, a_1 = 0$
we will take $\quad a_1 = 0,\ a_0 \neq 0$ and $t = \pm l$ to give the G.S.

$$\begin{aligned}
\text{coefficient of } x^{n+t} \quad & [(n+t)^2 - l^2]\, a_n + a_{n-2} = 0\\
\text{i.e.} \quad a_n &= -a_{n-2} \,/\, [(n+t)^2 - l^2]\\
&= -a_{n-2} \,/\, [(n+t+l)(n+t-l)]\\
\text{thus since } a_1 = 0, \quad & a_1 = a_3 = a_5 = a_7 \text{ etc } = 0
\end{aligned}$$

and for even powers, $a_2 = -a_0 \,/\, [(2+t+l)(2+t-l)]$

and $a_4 = a_0 \,/\, [(4+t+l)(4+t-l)(2+t+l+)(2+t-l)]$

Now the solution $y = x^t\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$

where $t = +l$ or $-l$

For $t = +l$
$$y = a_0\, x^l\, [1 - x^2/2(2l-2) + x^4/(2\cdot 4(2l-2)(2l+4)) - \cdots] \tag{37}$$

and if we give a_0 the conventional normalising value $a_0 = 1\,/\,(2^l\,\Gamma(l+1))$
then our series becomes a Bessel Function of the First Kind, order l, and is denoted by $J_l\,(x)$

For $t = -l$
$$y = b_0\, x^{-l}\, [1 - x^2/2(2-2l) + x^4/(2\cdot 4(2-2l)(4-2l)) - \cdots] \tag{38}$$

and if we give b_0 the conventional normalising value $b_0 = 1\,/\,(2^l\,\Gamma(1-l))$
then our series becomes a Bessel Function of the First Kind, order $-l$, and is denoted by $J_{-l}\,(x)$

If l is not an integer, the General Solution of Bessel's Equation is $y = A\,J_l(x) + B\,J_{-l}(x)$ ✓

But if l is an integer, for example if $l = -2$,

then (37) is $y = a_0\, x^{-2}\, [1 - x^2/2(2l-2) + x^4/(2\cdot 4(2l-2)(2l+4)) - \cdots]$

To avoid difficulties with $1\,/\,(2l+4)$,

we write $y = a_0/(2l+4)\cdot\, x^l\, [1 - x^2/2(2l-2) + x^4/(2\cdot 4(2l-2)(2l+4)) - \cdots]$

and call $a_0/(2l+4) = A$

putting $l = -2$, $y = A\, x^{-2}[x^4/(2\cdot 4\cdot -2)\; - \;x^6/(2\cdot 4\cdot 6\cdot -2\cdot 2) + \cdots]$

The second series (38) is simpler: putting $l = -2$, $y = B\, x^2\, [1 - x^2(2\cdot 6) + \cdots]$

But note that if $A = -16B$, the two series are identical !
Thus, robbed of one of our arbirtary constants, we cannot form the General Solution.

2 PARTIAL DIFFERENTIAL EQUATIONS

Or $f\;(x,\; y,\; z,\; \delta z/\delta x,\; \delta z/\delta y,\; \delta^2 z/\delta x^2,\; \delta^2 z/\delta y^2,\; \cdots)\; = \;0$

As in ordinary D.E.s, the order is defined as the order of the highest partial devivative in the equation.

A trivial example of a P.D.E.: $\delta z/\delta x = 0$

which has the solution $z = f(y)$ where $f(y)$ is an arbitrary function of integration.

2.1 Elimination of Arbitrary Constants

This can be used to form P.D.E.'s, just as with O.D.E.'s (section 1.2).

e.g. $z = ax + by$

differentiating $\delta z/\delta x = a$ and $\delta z/\delta y = b$

substituting $z = x\cdot\delta z/\delta x + y\cdot\delta z/\delta y$

2.2 Elimination of Arbitrary Functions

e.g.
$$z = x\cdot f(y/x) \tag{39}$$

differentiating δx
$$\delta z/\delta x = f(y/x) - (y/x)\cdot f'(y/x) \tag{40}$$

differentiating δy
$$\delta z/\delta y = (x/x)\cdot f'(y/x) \;=\; f'(y/x) \tag{41}$$

substituting (41) in (40) $\delta z/\delta x + (y/x)\delta z/\delta y = f(y/x)$

multiplying by x $x\cdot\delta z/\delta x + y\cdot\delta z/\delta y = x\cdot f(y/x)$

substituting in (39) $z = x\cdot\delta z/\delta x + y\cdot\delta z/\delta y$ ✓

or e.g.	$z = f(x-ay) + g(x+ay)$	where a is given
differentiating δx	$\delta z/\delta x = f'(x-ay) + g'(x+ay)$	
differentiating δy	$\delta z/\delta y = -a\, f'(x-ay) + a\, g'(x+ay)$	
and again	$\delta^2 z/\delta x^2 = f''(x-ay) + g''(x+ay)$	
and	$\delta^2 z/\delta y^2 = a^2 f''(x-ay) + a^2 g''(x+ay)$	
thus	$\delta^2 z/\delta y^2 = a^2\, \delta^2 z/\delta x^2$	✓

2.3 Linear Partial Differential Equation with Constant Coefficients

e.g. $\delta z/\delta x - \alpha\, \delta z/\delta y = 0$

consider $f(\alpha x + y)$, such that $\delta f/\delta x = \alpha f' = \alpha\, \delta f/\delta y$

thus $z = f(\alpha x + y)$ ✓

e.g. $\delta^2 z/\delta x^2 = 0$

integrating, $\delta z/\delta x = f(y)$

and again, $z = x\, f(y) + g(y)$ ✓

e.g. $\delta^2 z/\delta y^2 = xy$

integrating, $\delta z/\delta y = x^2 y/2 + f(y)$

and again,
$$z = x^2y^2/4 + \int f(y)dy + g(x)$$
$$= x^2y^2/4 + F(y)dy + g(x)$$ ✓

2.4 The Homogeneous Partial Differential Equation

This is $\delta^2 z/\delta x^2 + A\, \delta^2 z/\delta x \delta y + B\, \delta^2 z/\delta y^2 = \phi(x, y)$

As in section 1.10.1, we put $D = \delta/\delta x$ and $D' = \delta/\delta y$

so the equation becomes $(D^2 + ADD' + BD'^2)\, z = \phi(x, y)$

to solve, we factorise into $(D - \alpha D')\,(D - \beta D')\, z = \phi(x, y)$ where α and β are constants.

As before, the General Solution = Complimentary Function + a Particular Integral,
where the C.F. is the G.S. of $(D - \alpha D')\,(D - \beta D')\, z = 0$
and the proof is exactly analagous to that in section 1.10.1.

2.4.1 Finding the Complimentary Function

This is the general solution of $(D - \alpha D')\,(D - \beta D')\, z = 0$

which has two solutions, $D = \alpha D'$ and $D = \beta D'$

or $\delta z/\delta x - \alpha \delta z/\delta y = 0$ and $\delta z/\delta x - \beta \delta z/\delta y = 0$

We solved these in section 2.3: they give $z = f(\alpha x + y)$ and $z = g(\beta x + y)$

Therefore the sum $z = f(\alpha x + y) + g(\beta x + y)$ is also a solution,

and indeed **if** $\alpha \neq \beta$ it has two arbitrary functions, and is therefore the Complimentary Function.

But if $\alpha = \beta$, then we use Raimes' Rule: $z = f(\alpha x + y) + xg(\alpha x + y)$

or $z = f(\alpha x + y) + yg(\alpha x + y)$

or $z = xf(\alpha x + y) + yg(\alpha x + y)$

These are all equivalent General Solutions.

2.4.2 Finding the Particular Integral

The method of putting $z = 1/[(D-\alpha D')(D-\beta D')] \cdot \phi(x,y)$
and expanding in series, as in section 1.10.3, can still be used.

For example	$\delta^2 z/\delta x^2 - 3\delta^2 z/\delta x \delta y + 2\delta^2 z/\delta y^2 = xy$
or	$(D^2 - 3DD' + 2D'^2)\, z = xy$
i.e.	$(D-2D')(D-D')\, z = xy$
thus the C.F. is	$z = f(2x+y) + g(x+y)$
and a P.I. is	$z = 1/[(D-2D')(D-D')] \cdot xy$
	$= 1/[D^2(1-2D'/D)(1-D'/D)] \cdot xy$
	$= 1/D^2 \cdot (1+2D'/D+\cdots)(1+D'/D+\cdots) \cdot xy$
	$= 1/D^2 \cdot (1+3D'/D+\cdots) \cdot xy$
	$= 1/D^2 \cdot (xy + 3x/D + \cdots)$
	$= 1/D^2 \cdot (xy + 3x^2/2 + \cdots)$
	$= 1/D \cdot (x^2y/2 \ + \ x^3/2)$
	$= x^3y/6 \ + \ x^4/8$
thus G.S = C.F. + P.I. is	$z = f(2x+y) + g(x+y) + x^3y/6 + x^4/8$ ✓

2.5 Homogeneous Linear P.D.E. with Constant Coefficients

$$\frac{\delta^2 z}{\delta x^2} + A\frac{\delta^2 z}{\delta x \delta y} + B\frac{\delta^2 z}{\delta y^2} + C\frac{\delta z}{\delta x} + E\frac{\delta z}{\delta y} + Mz = \phi(x,y)$$

This is solved, like the homogeneous P.D.E., by factorising into $(D-\alpha D'-m)(D-\beta D'-n)z = \phi(x,y)$

This is not always possible, e.g. $\delta^2 z/\delta x^2 - \delta z/\delta y = 0$

because $(D^2 - D')\, z = 0$ is not factorisable since $\sqrt{D}'$ has no meaning

Then as before, General Solution = Complimentary Function + Particular Integral

2.5.1 Finding the Complimentary Function

This is, as before, the solution of the 'Reduced Equation'.

The Reduced Equation is	$(A-\alpha D'-m)\,(A-\beta D'-n) = 0$
which has two solutions	$A-\alpha D'-m = 0$ and $A-\beta D'-n = 0$
We know from section 2.3	if $n=0$ this gives $z = f(\beta x - y)$
so we try the substitution	$z = v(x)\cdot(\beta x - y)$
so that	$Dz = vDf + fDv$
and	$D'z = vD'f$
Substituting this in	$(A-\alpha D'-m) = 0$
gives	$vDf + fDv - \beta vD'f - nvf = 0$
but since $(D-\beta D')f = 0$, then	$vDf = \beta vD'f$
substituting,	$fDv - nvf = 0$
i.e.	$Dv - nv = 0$
i.e.	$dv/dx = nv$
this 1st-order D.E gives	$v = e^{nx}$
so that	$z = e^{nx}f(\beta x + y)$

and similarly for the other solution, so that the C.F. is $z = e^{mx}g(\alpha x + y) + e^{nx}f(\beta x + y)$
This fails to be the C.F. if $\alpha = \beta$ and $m = n$; then the C.F. is $z = e^{nx}f(\alpha x + y) + x\,e^{nx}g(\alpha x + y)$?

2.5.2 Finding the Particular Integral

As before, $z = [1/((D - \alpha D' - m)\,(D - \beta D' - n)] \cdot \phi(x,y)$

i.e. $z = (1/mn) \cdot [1/((1 - D/m - \alpha D'/m)\,(1 - D/n - \beta D'/n)] \cdot \phi(x,y)$

If $\phi(x,y)$ is a polynomial in x and y, then the expansion method of section 1.10.3 still works.
If $\phi(x,y)$ is not a polynomial, then we may still be able to find a particular integral, if it is of one of a number of special forms analagous to those in section 1.10.4.
For example, if $\phi(x,y) = e^{ax+by} \cdot u(x,y)$ where $u(x,y)$ is a polynomial,
then we can show that for any operator $F(D, D')$ which is a polynomial or series in D and D', then

$$
\begin{array}{lrl}
\text{then} & F(D,D')\,[e^{ax+by} \cdot u(x,y)] & = e^{ax+by} \cdot F(D+a, D'+b) \cdot u(x,y) \\
\text{Proof :} & D\,[e^{ax+by} \cdot u(x,y)] & = a e^{ax+by} \cdot u + e^{ax+by} \cdot Du \\
\text{i.e.} & & = e^{ax+by} \cdot (D+a) \cdot u \\
\text{similarly} & D'\,[e^{ax+by} \cdot u(x,y)] & = e^{ax+by} \cdot (D'+b) \cdot u \\
\text{re-applying,} & D^n\,(e^{ax+by} \cdot u) & = e^{ax+by} \cdot (D+a)^n \cdot u \\
\text{further} & D^m D'^n\,(e^{ax+by} \cdot u) & = e^{ax+by} \cdot (D+a)^m (D'+b)^n \cdot u \\
\text{Thus the polynomial} & F(D,D') & = \sum a_i\, D^m\, D'^n \\
\text{gives} & F(D,D') \cdot e^{ax+by} \cdot u & = \sum a_i\, D^m\, D'^n\, e^{ax+by} \\
 & & = e^{ax+by} \cdot \sum a_i\, D^m\, D'^n \cdot u \\
 & & = e^{ax+by} \cdot F(D,D') \cdot u \qquad Q.E.D.
\end{array}
$$

Hence, we can also prove that $(1/F(D,D')) \cdot e^{ax+by} \cdot u = e^{ax+by} \cdot (1/F(D+a, D+b))) \cdot u$
either by expanding in series, or by multiplying both sides on the left by $F(D, D')$

With these theorems we can solve the case $\phi(x,y) = e^{ax+by} \cdot (polynomial)$

$$
\begin{array}{ll}
\text{For example,} & (D - D')(D + D' + 1)\, z = x\, e^{x+y} \\
\text{the C.F. is} & z = e^{mx} g(\alpha x + y) + e^{nx} f(\beta x + y) \\
\text{or in this case} & \quad = f(x+y) + e^{-x} g(-x+y) \\
\text{The P.I. is} & z = [1/((D - D)\,(D + D' + 1)] \cdot x\, e^{x+y} \\
 & z = e^{x+y} \cdot [1/((D - D)\,(D + D' + 3)] \cdot x \\
\text{now } D'x = 0, & \text{thus we can put } D' = 0 \\
\text{giving} & z = e^{x+y}\,[1/D(D+3)] \cdot x \\
 & \quad = e^{x+y}\,(1/3D)\,(1 - D/3 + \cdots) \cdot x \\
 & \quad = e^{x+y}\,(1/3D)\,(x - 1/3) \\
 & \quad = (1/3)\, e^{x+y}\,(x^2/2 - x/3) \\
\text{Thus G.S. = C.F. + P.I.} & \quad = f(x+y) + e^{-x} \cdot g(-x+y) + (e^{x+y}/3)(x^2 - x/3) \qquad \checkmark
\end{array}
$$

Similarly, by taking real parts, as in section 1.10.4,
we can integrate the case $\phi(x,y) = cos(ax+by) \cdot u(x,y)$ if $u(x,y)$ is a polynomial.

$$
\begin{array}{lll}
\text{The P.I. is} & z = [1/F(D,D')] \cdot cos(ax+by) \cdot u(x,y) & \\
 & \quad = \Re\,[1/F(D,D')] \cdot e^{i(ax+by)} \cdot u(x,y) & \\
 & \quad = \Re\, e^{i(ax+by)}\,[1/F(D+ia, D'+ib)] \cdot u(x,y) & \\
\text{e.g. Find a P.I. of} & z = [1/(D^2 + D')^2]\, cos(x - 2y) & \text{i.e. } u = 1 \\
 & \quad = \Re\, e^{i(x-2y)}\,[1/((D+i)^2 + (D' - 2i)^2)]\, 1 & \text{but } D \cdot 1 = 0 \text{ and } D' \cdot 1 = 0 \\
\text{put } D = 0 \text{ and } D' = 0 & z = \Re\, e^{i(x-2y)}/-5 & \\
 & \quad = -1/5 \cdot cos(x - 2y) & \checkmark
\end{array}
$$

2.6 Separable Solutions

Some equations have solutions of the form $z = X(x) \cdot Y(y)$

so that
$$\delta z/\delta x = X'Y$$
$$\delta z/\delta y = XY'$$
$$\delta^2 z/\delta x^2 = X''Y$$
$$\delta^2 z/\delta y^2 = XY''$$
and
$$\delta^2 z/\delta x \delta y^2 = X'Y'$$

Many very important equations in Physics have such solutions, for example

the Wave Equation $\quad \delta^2 z/\delta x^2 = a^2\, \delta^2 z/\delta y^2 \quad$ where y = time

Laplace's Equation $\quad \delta^2 z/\delta x^2 + \delta^2 z/\delta y^2 = 0$

the Diffusion Equation $\quad \delta^2 z/\delta x^2 = a^2\, \delta z/\delta y \quad$ where y = time

2.7 The Wave Equation

the Wave Equation is $\quad \delta^2 z/\delta x^2 = a^2\, \delta z/\delta y$

putting $z = X(x) \cdot Y(y)$ $\quad X''Y = a^2\, XY''$

dividing by za^2 $\quad X''/a^2 X = Y''/Y$

Now (this is the crucial trick) since the LHS is a function of x only, and the RHS is a function of y only, and x and y are independent, then LHS and RHS must both be equal to the same constant !
This is known as the "separation constant".
As we can obtain different solutions if the constant is positive or negative, we put the constant $= \pm k^2$

1) separation constant $\quad = +k^2$

then $\quad X'' = a^2\, k^2\, X$ and $Y'' = k^2\, Y$

i.e. $\quad X = A\cosh(akx) + B\sinh(akx)$ and $Y = C\cosh(kx) + D\sinh(kx)$

so that $\quad z = X \cdot Y = [A\cosh(akx) + B\sinh(akx)] \cdot [C\cosh(kx) + D\sinh(kx)]$

2) separation constant $\quad = -k^2$

then $\quad X'' = -a^2\, k^2\, X$ and $-Y'' = k^2\, Y$

so that $\quad z = X \cdot Y = [A\cos(akx) + B\sin(akx)] \cdot [C\cos(kx) + D\sin(kx)]$

2.8 Laplace's Equation

Laplace's Equation is $\delta^2 z/\delta x^2 + \delta^2 z/\delta y^2 = 0$ so we can just put $a = i$ in the wave equation;
using $\cos(ikx) = \cosh(kx)$, $\cosh(ikx) = \cos(kx)$
and $\sin(ikx) = \sinh(kx)$, $\sinh(ikx) = \sin(kx)$
gives us the two solutions :

1) $\quad z = (A\cos\, kx + B\sin\, kx) \cdot (C\cosh\, ky + D\sinh\, ky)$

2) $\quad z = (A\cosh\, kx + B\sinh\, kx) \cdot (C\cos\, ky + D\sin\, ky)$

2.9 The Diffusion Equation

the Diffusion Equation is $\quad \delta^2 z/\delta x^2 = a^2\, \delta z/\delta y \quad$ where y = time

putting $z = X(x) \cdot (Y(y)$, $\quad X''Y = a^2\, XY'$

divide by $a^2 z$, $\quad X''/a^2 X = Y'/Y = constant$

$\quad = \pm k^2 \quad$, say

1) separation constant $= +k^2$

then $X'' = a^2\, k^2\, X$ and $Y' = k\, Y$

giving $X = A \cosh(akx) + B \sinh(akx)$ and $Y = C \exp(k^2 y)$

so that $z = X \cdot Y = [A \cosh(akx) + B \sinh(akx)] \cdot \exp(k^2 y)$

2) separation constant $= -k^2$

then $X'' = -a^2\, k^2\, X$ and $-\, Y'' = -k^2\, Y$

giving $z = X \cdot Y = [A \cos(akx) + B \sin(akx)] \cdot \exp(-k^2 y)$

2.10 Boundary Conditions

1) e.g.: Solve Laplace's equation, given $x = 0$ when $x = 0$, $y = 0$ and when $x = \pi$, $y = 0$
Taking a hint from the π, we will use the form of solution which is trigonometric in x, ie:

$$z = (A \cos\, kx + B \sin\, kx) \cdot (C \cosh\, ky + D \sinh\, ky)$$

from $z = 0$ when $x = 0$, we have $A = 0$

from $z = 0$ when $y = 0$, we have $C = 0$

thus $z = b\, \sin(kx) \cdot \sinh(ky)$

from $z = 0$ when $x = \pi$, we have $\sin\, kx = 0$

whence $k = n$ where k is an integer ✓

2) e.g.: Solve Laplace's equation, given $z = 0$ when $x = 0$ or $x = \pi$, and $\delta z/\delta y = 0$ when $y = 0$
The conditions on x are the same as in the previous example,

thus $z = B\, \sin(nx) \cdot (C \cosh(ny) + D \sinh(ny))$

so that $\delta z/\delta y = nB\, \cos(nx) \cdot (nC \sinh(ny) + nD \cosh(ny))$

but from $\delta z/\delta y = 0$ when $y = 0$, we have $D = 0$

so $z = b\, \sin(nx) \cdot \cosh(ny)$ ✓

3) e.g.: Solve Laplace's equation, given $z = 0$ when $x = 0$ or $x = \pi$, and $z \to 0$ as $y \to \infty$

using $z = (A \cos\, kx + B \sin\, kx) \cdot (C\, e^{ky} + D\, e^{-ky})$

assuming $z = 0$ when $x = 0$, then from $z \to 0$ as $y \to \infty$, we have $C = 0$

also from $z = 0$ when $x = 0$, we have $A = 0$

therefore $z = B\, \sin(kx) \cdot e^{-ky}$

also, from $z = 0$ when $x = \pi$, k must be a positive integer.

thus $z = B\, \sin\, nx \cdot e^{-ny}$ with $(n > 0)$ ✓

Now suppose that in this problem, we impose the extra boundary condition $z = 1$ when $y = 0$
In $z = B\, \sin\, nx \cdot e^{-ny}$ this would mean $B\, \sin\, nx = 1$ which is impossible.
But what we can do is construct the solution $z = \sum_1^n B_n\, \sin\, nx \cdot e^{-ny}$
Then, from $z = 1$ when $y = 0$, we have $1 = \sum_1^n B_n\, \sin\, nx$
and if we are only interested in the range $1 \le x \le \pi$ we can use the fact that

$$1 = \sum 2/(\pi n) \cdot [1 - (-1)^n]\, \sin(nx) \qquad \text{for } 1 \le x \le \pi$$

thus $z = \sum 2/(\pi n) \cdot [1 - (-1)^n]\, \sin(nx) \exp(-ny)$ ✓

$$= 4/\pi \cdot [\, e^{-y} \sin x + (e^{-3y} \sin 3x)/3 + (e^{-5y} \sin 3x)/5 \cdots]$$

This is not valid for all y $z = \sum 2/(\pi n) \cdot [1 - (-1)^n]\, \sin(nx) \exp(-ny)$

since $\delta z/\delta x = \sum 2/\pi \cdot [1 - (-1)^n]\, \cos(nx) \exp(-ny)$

and $\delta^2 z/\delta x^2 = \sum 2n/\pi \cdot [(-1)^n - 1]\, \sin(nx) \exp(-ny)$

and $\delta z/\delta y = \sum 2/\pi \cdot [(-1)^n - 1]\, \sin(nx) \exp(-ny)$

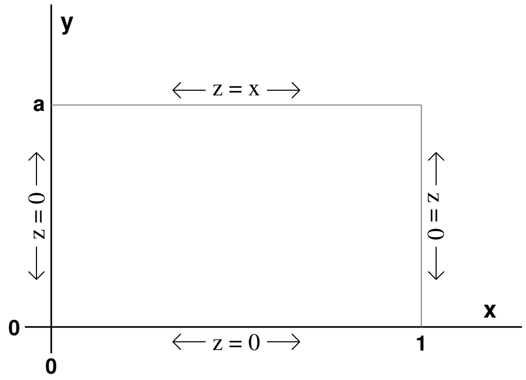
and $\delta^2 z/\delta y^2 = \sum 2n/\pi \cdot [1 - (-1)^n]\, \sin(nx) \exp(-ny)$

All these series diverge if $y < 0$, but they all converge if $y > 0$.
Thus our solution is valid in the domain $0 \le x \le \pi$ and $y > 0$

4) e.g.: Find a solution of Laplace's equation,

valid in the region	$0 \le x \le \pi$ and $0 \le y \le a$
given	$z = 0$ when $x = 0$ or $x = \pi$,
and	$z = 0$ when $y = 0$
and	$z = x$ when $y = a$ for $0 \le x \le \pi$.

This is then the region:



Laplace's Equation is $\delta^2 z/\delta x^2 + \delta^2 z/\delta y^2 = 0$

therefore	$z = (A\cos kx + B\sin kx)\cdot(C\cosh ky + D\sinh ky)$	see section 2.8
from	$z = 0$ when $x = 0$, then $A = 0$	
from	$z = 0$ when $y = 0$, then $C = 0$	
thus	$z = B\sin kx \cdot \sinh ky$	
from	$z = 0$ when $x = \pi$, then $\sin k\pi = 0$, so $k = n$	
thus	$z = B\sin nx \cdot \sinh ny$	
But now for	$z = x$ when $y = a$	
we put	$z = \sum B_n \sin nx \cdot \sinh ny$	
giving	$z = \sum B_n \sin nx \cdot \sinh na$	
but using	$\sum(2/n)(-1)^{n+1}\sin nx = x$	for $-\pi < x < \pi$
we get	$B_n \sinh(na) = (2/n)(-1)^{n+1}$	
giving	$z = 2\sum(-1^{n+1}/n)\cdot\sin(nx)\sinh(ny) / \sinh(na)$	✓

This solution is in fact valid for $-\pi < x < 0$, as well as in the region $0 < x < \pi$
And as far as y is concerned, $\sinh(ny)/\sinh(na) = (e^{ny} - e^{-ny}) / (e^{na} - e^{-na})$
which for large n becomes $e^{n(y-a)}$ if $y > 0$
and if $y > a$, this diverges to infinity with large n ; thus $0 < y < a$

5) e.g.: A thin rod of unity length is at 1°C. Then the two ends are plunged into ice.
The rod's thermal diffusivity is 1.

So :	$\delta\Theta/\delta y = 1 \cdot \delta^2\Theta/\delta t^2$	
and we must find	$\Theta(x,t)$	
At the ends,	$\Theta = 0$ when $x = 0$ or $x = 1$	for all t
Initially,	$\Theta = 1$ when $t = 0$	for $0 < x < 1$

Now the solution of the Diffusion Equation is $z = (A\cos kx + B\sin kx)\cdot\exp(-k^2 y)$ (from section 2.9)
where we have taken the '$-k^2$' solution, so that it decays as $t \to \infty$

From $\Theta = 0$ when $x = 0$ then $A = 0$

giving $\Theta = B \sin kx \exp -k^2 t$

from $\Theta = 0$ when $x = 1$ then $k = n\pi$

giving $\Theta = B \sin(n\pi x) \exp(-n^2\pi^2 t)$

but for $\Theta = 1$ when $t = 0$ we will have to

form the series $\Theta = \sum B_n \sin(n\pi x) \exp(-n^2\pi^2 t)$

whence $1 = \sum B_n \sin(n\pi x)$ for $0 < x < 1$

thus, $B_n = \int_0^1 1 \cdot \sin(n\pi x)\, dx$ by Fourier

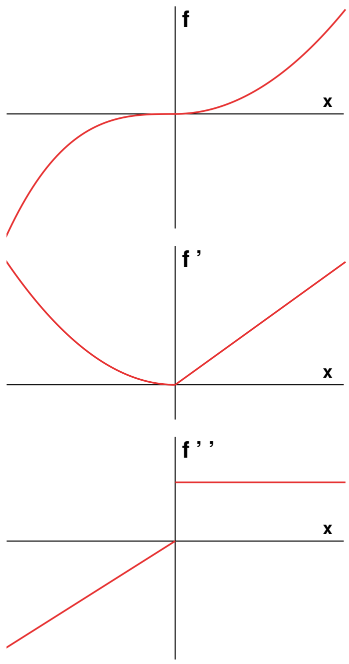
$$= (-2/n\pi)\,[\cos((n\pi x)\,]_0^1$$
$$= (-2/n\pi)\,((-1)^n - 1)$$
$$= (2/n\pi)\,(1 - (-1)^n)$$

substituting, $\Theta = (2/\pi) \sum (1/n)\,(1 - (-1)^n) \sin(n\pi x) \exp(-n^2\pi^2 t)$ ✓

or $\Theta = (4/\pi)\,[\sin(\pi x) \exp(-\pi^2 t) + (1/3) \sin(3\pi x) \exp(-9\pi^2 t) + \cdots]$

Note that in, practice, the higher modes decay very rapidly.

2.11 Higher Derivatives



The derivative $f' = d/dx\,[f(x)] = \lim_{h\to 0}(f(x+h) - f(x))\,/\,h$
exists only if this limit is well-defined.
Many functions possess derivatives only up to a certain order. E.g.:

$$f(x) = \begin{cases} x^3 & \text{for } x \le 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$$

is continuous. Moreover, it is differentiable:

$$f'(x) = \begin{cases} 3x^2 & \text{for } x \le 0 \\ 2x & \text{for } x \ge 0 \end{cases}$$

But f' is not differentiable:

$$f''(x) = \begin{cases} 6x & \text{for } x \le 0 \\ 2 & \text{for } x \ge 0 \end{cases}$$

which is undefined at $x = 0$.

2.12 Leibnitz's Theorem

Leibnitz's Theorem concerns the n-th order derivative of a product.

For example: $d(uv)/dx = (du/dx) \cdot v + u \cdot (dv/dx)$

$$d^2(uv)/dx^2 = (du^2/dx^2)v + 2\,(du/dx)(dv/dx) + u(d^2v/dx^2)$$
$$d^3(uv)/dx^3 = (du^3/dx^3)v + 3\,(du^2/dx^2)(dv/dx^2) + 3\,(du/dx)(dv^2/dx^2) + u(d^3v/dx^3)$$

Obviously there is something like the Binomial Theorem going on.
Let us simplify our notation: put $d^n f/dx^n \equiv f_n$
Assuming that u and v are both n times differentiable, we wish to prove

$$(uv)_n = u_n v + n u_{n-1} v_1 + (n(n-1)/2) u_{n-2} v_2 + \cdots + n u_1 v_{n-1} + u v_n$$

or $$(uv)_n = \sum_0^n \binom{n}{r} u_{n-r}\, v_r$$ ✓

where $$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$\text{First we need to evaluate:} \quad \binom{n}{r} + \binom{n}{r-1} = \frac{n(n-1)\cdots(n-r+1)}{r!} + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}$$
$$= \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}\left(\frac{n-r+1}{r}+1\right)$$
$$= \frac{(n+1)n(n-1)\cdots(n-r+2)}{(r-1)!}$$
$$= \binom{n+1}{r}$$

Now we prove the theorem by induction. First note that it is true for $n=1:\quad (uv)_1 = u_1v + uv_1$
So we assume that it is true for $n=m$, and then differentiate to evaluate for $m+1$ ...

$$(uv)_m = u_mv + mu_{n-1}v_1 + m(m-1)u_{m-2}v_2/2! + \cdots + mu_1v_{m-1} + uv_m$$
$$(uv)_{m+1} = (u_{m+1}v + u_mv_1) + \binom{m}{1}(u_mv_1 + u_{m-1}v_2) + \binom{m}{2}(u_{m-1}v_2 + u_{m-2}v_3)\cdots + (u_1v_m + uv_{m+1})$$
$$= u_{m+1}v + \left[1+\binom{m}{1}\right]u_mv_1 + \left[\binom{m}{1}+\binom{m}{2}\right]u_{m-1}v_2 + \left[\binom{m}{2}+\binom{m}{3}\right]u_{m-2}v_3\cdots + uv_{m+1}$$
$$= u_{m+1}v + \binom{m+1}{1}u_mv_1 + \binom{m+1}{2}u_{m-1}v_2 + \cdots + uv_{m+1}$$

so we see that it is also true for $m+1$. We have thus proved Leibnitz's Theorem by induction.

For example: $$d^5(x^4e^x)/dx^5 = u_5v + 5u_4v_1 + 10u_3v_2 + 10u_2v_3 + 5u_1v_4 + uv_5$$
$$= e^x\left(120 + 240x + 120x^2 + 20x^3 + x^4\right) \qquad \checkmark$$

2.13 Application of Leibnitz's Theorem to Differential Equations

Certain Differential Equations can be solved using Leibnitz's Theorem.
For example, Bessel's Equation of order zero: $xy'' + y' + xy = 0$, or $xy_2 + y_1 + xy = 0$

Obtain a series solution such that $y=1$ and $y'=0$ when $x=0$
so we set our solution as: $y(x) = y(0) + xy_1(0) + x^2y_2(0)/2! + x^3y_3(0)/3! + \cdots$
and differentiate Bessel's Equation n times : $xy_{n+2} + ny_{n+1} + y_{n+1} + xy_n + ny_{n-1} = 0$
Therefore to find y_n for $x=0$, we put $x=0$

i.e. $\quad n\,y_{n+1}(0) + y_{n+1}(0) + n\,y_{n-1}(0) = 0$
or $\quad y_{n+1}(0) = -n/(n+1)\cdot y_{n-1}(0)$
But we know $\quad y(0) = 1$ and $y_1(0) = 0$
thus $\quad y_2(0) = -y(0)/2 = -1/2$ and $y_3 = 0$
$\quad y_4(0) = -3\,y_2(0)/4 = -3/4\cdot 1/2$ and $y_5 = 0$
$\quad y_6(0) = -5\,y_4(0)/6 = -5/6\cdot 3/4\cdot 1/2$
our solution is $\quad y = 1-(1/2)x^2/2! + (1/2)(3/4)x^4/4! - (1/2)(3/4)(5/6)x^6/6! + \cdots$
$\quad = 1 - x^2/2^2 + x^4/(2^2.4^2) - x^6/(2^2.4^2.6^2) + \cdots \qquad \checkmark$

which is indeed a Bessel function of the 1st kind, order zero.
If the right-hand side of the equation had been not zero, but a polynomial of order i,
we would normally handle this by differentiating $n+3$ ($n+i$?) times, instead of n times.